NOTE

Analysis of the Error for Approximations to Systems of Hyperbolic Equations

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1. INTRODUCTION

The Yee scheme [7, 2] is one of the basic algorithms used in solving the time dependent Maxwell equations. This is a second order accurate method based on central differences in space and time and a staggered mesh. In order to construct a similar method based on a colocated grid it is necessary to represent a central second difference as a product of forward and backward first order differences. Rather than successively applying forward and backward differences one can split the system and apply forward differences to some components and backward differences to other components without accounting for characteristic information. Liu [3, 1] already used this idea to construct a modification of the Yee scheme for both Cartesian and general grids. Converting these approximations for the system to a scalar wave equation both the schemes of Yee and Liu yield the standard second order approximation to the wave equation. Liu already noted that the numerical wave speed of both systems is identical. Hence, Liu concluded that his scheme is second order accurate. However, a Taylor series analysis of the Liu scheme indicates that it is only first order accurate in the L_2 sense. Scheme II studied in the next section is a simplified version of the Liu algorithm.

For higher accuracy we can replace the Yee scheme with a fourth order implicit scheme for the Maxwell equations based on the same staggered mesh as the Yee scheme [8, 5]. This is presented later as Scheme III. When the equivalent scheme is derived for the scalar second order equation it is not as compact as possible. Therefore, we attempted to find an algorithm for the first order system that was equivalent to the most compact implicit fourth order accurate scheme for the wave equation. This again requires using forward differences on some equations and backward differences for other equations. The simplification of this

scheme to two equations in one space dimension is later presented as Scheme IV. However, Scheme IV is not fourth order accurate even though the first order system and the second order scalar equation are seemingly equivalent.

The Maxwell equations are usually solved as a first order system rather than a second order equation because of difficulties with inhomogeneous media and with boundary conditions. Hence, we concentrate on the properties of the first order system.

Therefore, the purpose of this investigation is to resolve the following seeming paradox. The wave equation may be expressed as either a scalar second order equation or a system of first order equations. Both sets can be approximated by finite difference methods. Any finite difference scheme for a first order system can be recast as an algorithm for a second order scalar partial differential equation. One would expect that the order of accuracy of both approaches would be the same. It turns out, however, that this is not necessarily true. Rather, the approximation to the first order system may be of lower order accuracy, in L_2, than the apparently equivalent approximation to the scalar second order equation. To resolve this apparent paradox we study the details of the amplutude and phase errors together with the initialization of the scheme for a system of equations. The difficulties that we analyze are peculiar to a system and it cannot be reduced to the study of a scalar first order hyperbolic equation. In this investigation we shall analyze only the pure initial value problem. Since the Maxwell equations, for example, are cast as a first order system we endeavor to provide pre- and post-processing that resolves this paradox, i.e., they preserve the higher order accuracy for the first order system. This pre/post-processing is not necessary for the scalar wave equation.

In Section 2 we examine the one dimensional problem. It is shown that this seeming paradox affects only the amplitude of the numerical solution. The phase of both sets of approximations is the same. We trace the source of the ambiguity to the different treatments of the initial conditions in the two formulations. In Section 3 we show how to recover the full accuracy for the system formulation by employing pre- and post-processing of the data. In Section 4 we extend the results to the two dimensional case.

2. ONE DIMENSION

2.1. *Systems of Equations*

We first examine the concept of order of accuracy for the simplest one dimensional system,

$$p_t = q_x$$
$$q_t = p_x \tag{1}$$

which is equivalent to the second order equation

$$p_{tt} = p_{xx}, \qquad q_{tt} = q_{xx}.$$

We first consider the one dimensional form of the Yee scheme used in computational electromagnetics, see [4]. We denote the numerical approximation to p and q by u and v, respectively,

Yee Scheme

$$u_j^{n+1} - u_j^n = \frac{\Delta t}{\Delta x}\left(v_{j+1/2}^{n+1/2} - v_{j-1/2}^{n+1/2}\right)$$
$$v_{j+1/2}^{n+3/2} - v_{j+1/2}^{n+1/2} = \frac{\Delta t}{\Delta x}\left(u_{j+1}^{n+1} - u_j^{n+1}\right). \tag{2}$$

Each of the above equations is second order accurate in space and time and hence the system is second order accurate. Consider the following variation of this scheme

Scheme II

$$u_j^{n+1} - u_j^n = \frac{\Delta t}{\Delta x}\left(V_j^n - V_{j-1}^n\right)$$

$$V_j^{n+1} - V_j^n = \frac{\Delta t}{\Delta x}\left(u_{j+1}^{n+1} - u_j^{n+1}\right).$$

(3)

The two schemes are not that different. Indeed, if we identity $V_j^n = v_{j+1/2}^{n+1/2}$ they are identical. Another way of seeing this is by reducing systems (2) and (3) to a second order equation approximation of the wave equation. We get in both cases the same second order central difference algorithm

$$u_j^{n+1} - 2u_j^n + u_j^{n-1} = \left(\frac{\Delta t}{\Delta x}\right)^2 \left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right).$$

(4)

However, since each equation in (3) is only first order accurate Scheme II is not second order accurate! The purpose of this paper is to analyze the source of this phenomenon and suggest ways of resolving it.

We now consider a more complicated case. Define $\Delta u = u^{n+1} - u^n$ and $\Delta v = v^{n+3/2} - v^{n+1/2}$ and consider [5]

Scheme III

$$\alpha(\Delta u_{j-1} + \Delta u_{j+1}) + (1 - 2\alpha)\Delta u_j = \frac{\Delta t}{\Delta x}\left(v_{j+1/2}^{n+1/2} - v_{j-1/2}^{n+1/2}\right)$$

$$\alpha(\Delta v_{j-1/2} + \Delta v_{j+3/2}) + (1 - 2\alpha)\Delta v_{j+1/2} = \frac{\Delta t}{\Delta x}\left(u_{j+1}^{n+1} - u_j^{n+1}\right).$$

(5)

Each of the steps in (5) is fourth order accurate if $\alpha = \frac{1}{24}$. Reducing Scheme III to a second order equation we obtain the following fourth order accurate scheme,

$$\alpha^2 \Delta^2 (u_{j+2} + u_{j-2}) + 2\alpha(1 - 2\alpha)\Delta^2 (u_{j+1} + u_{j-1}) + (1 - 4\alpha + 6\alpha^2)\Delta^2 u_j$$

$$= \left(\frac{\Delta t}{\Delta x}\right)^2 \left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right).$$

(6)

Though this is a fourth order scheme it is not compact since it uses the points u_{j+2} and u_{j-2}. We can achieve a compact implicit approximation to the wave equation by

$$\beta\Delta^2 (u_{j+1} + u_{j-1}) + (1 - 2\beta)\Delta^2 u_j = \left(\frac{\Delta t}{\Delta x}\right)^2 \left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right).$$

(7)

This scheme is fourth order accurate in space (and second order accurate in time) when $\beta = \frac{1}{12}$. We wish to rewrite (7) as a first order system. This can be accomplished by

Scheme IV

$$\gamma\Delta u_{j+1} + (1 - \gamma)\Delta u_j = \frac{\Delta t}{\Delta x}\left(v_{j+1/2}^{n+1/2} - v_{j-1/2}^{n+1/2}\right)$$

$$\gamma\Delta v_{j-1/2} + (1 - \gamma)\Delta v_{j+1/2} = \frac{\Delta t}{\Delta x}\left(u_{j+1}^{n+1} - u_j^{n+1}\right).$$

(8)

We now face the same question we previously asked. Each equation in Scheme IV is first order accurate while the system which is supposedly equivalent to (7) should be fourth order accurate when $\gamma = (1 \pm \sqrt{1 - 4\beta})/2$.

2.2. Analysis

We shall only consider initial value problems in periodic domains (period $= 1$). We denote the Fourier (in space) transform of the exact solution of (1) by (P, Q). Then

$$
\begin{aligned}
P(k, t) &= A_{exact}e^{ikt} + B_{exact}e^{-ikt} \\
Q(k, t) &= C_{exact}e^{ikt} + D_{exact}e^{-ikt},
\end{aligned}
\tag{9}
$$

where k is the Fourier variable. Here

$$
\begin{aligned}
A_{exact} &= \frac{1}{2}(P_0 + Q_0), & B_{exact} &= \frac{1}{2}(P_0 - Q_0) \\
C_{exact} &= \frac{1}{2}(Q_0 + P_0), & D_{exact} &= \frac{1}{2}(Q_0 - P_0).
\end{aligned}
\tag{10}
$$

To simplify the analysis we consider the semi-discrete approximation to (1),

$$
\begin{aligned}
u_t &= D_+ v \\
v_t &= D_- u,
\end{aligned}
\tag{11}
$$

where D_+, D_- are general finite difference operators. We Fourier transform (11) in space to obtain

$$
\begin{aligned}
U_t &= \epsilon(\theta)V \\
V_t &= \mu(\theta)U.
\end{aligned}
\tag{12}
$$

The quantities $\epsilon(\theta)$ and $\mu(\theta)$ are the Fourier symbols of the operators D_+, D_- in (11) with $\theta = k\Delta x$. By consistency they both approximate the quantity ik up to some order. The characteristic variables are

$$
\sqrt{\mu}U \pm \sqrt{\epsilon}V.
$$

In general $\epsilon \neq \mu$, even to the order of the scheme, and so the system cannot be diagonalized in physical space without using pseudo-difference operators. We will denote by λ^2 the symbol corresponding to the approximation of the spatial second derivative obtained by $\lambda^2 = \epsilon\mu$. This approximates $-k^2$ up to some order.

The system (12) can be reduced to the second order equation

$$
U_{tt} = \lambda^2(\theta)U, \qquad V_{tt} = \lambda^2(\theta)V.
\tag{13}
$$

As examples, for the schemes previously presented we have

Scheme I

$$
\epsilon = \mu = \frac{2i\sin(\theta/2)}{\Delta x}
$$

$$
\lambda^2 = -\frac{4\sin^2(\theta/2)}{(\Delta x)^2}
$$

Scheme II

$$\epsilon = \frac{1 - e^{-i\theta}}{\Delta x}$$

$$\mu = \frac{e^{i\theta} - 1}{\Delta x}$$

$$\lambda^2 = -\frac{4 \sin^2(\theta/2)}{(\Delta x)^2}$$

Scheme III

$$\epsilon = \mu = \frac{2i \sin(\theta/2)}{\Delta x (1 - 4\alpha \sin^2(\theta/2))}$$

$$\lambda^2 = -\frac{4 \sin^2(\theta/2)}{(\Delta x)^2 (1 - 4\alpha \sin^2(\theta/2))^2}$$

$$\alpha = \frac{1}{24}$$

Scheme IV

$$\epsilon = \frac{2i \sin(\theta/2)}{\Delta x (\gamma e^{i\theta} + 1 - \gamma)}$$

$$\mu = \frac{2i \sin(\theta/2)}{\Delta x (\gamma e^{-i\theta} + 1 - \gamma)}$$

$$\lambda^2 = -\frac{4 \sin^2(\theta/2)}{(\Delta x)^2 (1 - 2\beta \sin^2(\theta/2))}$$

$$\gamma = \frac{1 \pm \sqrt{1 - 4\beta}}{2}, \qquad \beta = \frac{1}{12}.$$

We rephrase our question in the following way. For Scheme II, ϵ and μ are a first order approximation to the (spatial) first derivative (ik), whereas λ^2 is a second order approximation to the symbol of the second derivative, $(-k^2)$. Similarly, in Scheme IV, ϵ and μ represent the symbols of first order approximations to the first derivative while λ is a fourth order approximation to ik. To get more insight about the order of accuracy in the system (12) versus the scalar second order equation (13) we examine the solutions of these equations. Note that for all cases given above λ^2 is negative and so λ is pure imaginary. The general solution of the system (12) is

$$
\begin{aligned}
U(\theta, t) &= A e^{\lambda t} + B e^{-\lambda t}, \qquad \lambda = \lambda(\theta) \\
V(\theta, t) &= C e^{\lambda t} + D e^{-\lambda t}.
\end{aligned}
\tag{14}
$$

Given the initial conditions

$$U(\theta, 0) = U_0 = P_0, \qquad V(\theta, 0) = V_0 = Q_0$$

we determine A, B, C, D as

$$A = \frac{1}{2}\left(U_0 + \sqrt{\frac{\epsilon}{\mu}}V_0\right), \qquad B = \frac{1}{2}\left(U_0 - \sqrt{\frac{\epsilon}{\mu}}V_0\right)$$

$$C = \frac{1}{2}\left(V_0 + \sqrt{\frac{\mu}{\epsilon}}U_0\right), \qquad D = \frac{1}{2}\left(V_0 - \sqrt{\frac{\mu}{\epsilon}}U_0\right). \tag{15}$$

Comparing this with (10) we see that (15) is a first order approximation to (10) unless $\epsilon(\theta) = \mu(\theta)$ as occurs in Schemes I and III. Hence, the amplitude of the approximation is only first order accurate. However, since λ is a second order approximation to ik the phase is second order accurate.

3. RECOVERING THE AMPLITUDE AND PHASE ACCURACY

We have shown in the previous section that using one sided operators in each equation of the system (11) can preserve the phase accuracy of the original approximation but the amplitude is only first order accurate. To rectify this situation we preprocess the data, i.e., we replace the given initial conditions U_0, V_0 by new initial conditions W_0, Z_0 to be determined. Then (15) becomes

$$A = \frac{1}{2}\left(W_0 + \frac{\epsilon}{\lambda}Z_0\right), \qquad B = \frac{1}{2}\left(W_0 - \frac{\epsilon}{\lambda}Z_0\right)$$

$$C = \frac{1}{2}\left(Z_0 + \frac{\mu}{\lambda}W_0\right), \qquad D = \frac{1}{2}\left(Z_0 - \frac{\mu}{\lambda}W_0\right).$$

One choice for W_0, Z_0 is

$$W_0 = U_0, \qquad Z_0 = \frac{\lambda}{\epsilon}V_0. \tag{16}$$

With this choice

$$A = \frac{1}{2}(U_0 + V_0), \qquad B = \frac{1}{2}(U_0 - V_0)$$

$$C = \frac{\lambda}{2\epsilon}(V_0 + U_0), \qquad D = \frac{\lambda}{2\epsilon}(V_0 - U_0).$$

After preprocessing V_0 we see that A and B (i.e., U) are higher order accurate but C and D (i.e., V) are still only first order accurate. The result of solving (11) with initial conditions (16) is

$$W(\theta, t) = \frac{1}{2}(U_0 + V_0)e^{\lambda t} + \frac{1}{2}(U_0 - V_0)e^{-\lambda t}$$

$$Z(\theta, t) = \frac{\lambda}{2\epsilon}\left\{(V_0 + U_0)e^{\lambda t} + (U_0 - V_0)e^{-\lambda t}\right\}. \tag{17}$$

Now W approximates P with the desired order of accuracy, while Z approximates Q with only first order accuracy. However, we note that $\frac{\epsilon}{\lambda}Z$ is a higher order approximation of Q.

This suggests that we post-process Z by $\frac{\epsilon}{\lambda}$, i.e., we choose Y at the final time such that

$$Y(\theta, t) = \frac{\epsilon}{\lambda} Z(\theta, t) \tag{18}$$

and so (compare with (9), (10))

$$Y(\theta, t) = \frac{1}{2}(V_0 + U_0)e^{\lambda t} + \frac{1}{2}(V_0 - U_0)e^{-\lambda t}.$$

We shall shortly describe how to perform the pre- and post-processing in physical space.

For Scheme II the amplitude is exact and the entire error is due to the phase error. In a more general context one can construct schemes for which the numerical phase has a higher degree of accuracy than the amplitude [6]. For a dissipative scheme, e.g., using upwind methods, both the amplitudes and phases will have errors.

We have performed this adjustment of the initial and final data for the general system of two equations. We now demonstrate this explicitly for the above examples and show how to perform the pre- and post-processing in physical space. For Scheme II we have

$$\epsilon = \frac{1 - e^{-i\theta}}{\Delta x}, \qquad \mu = \frac{e^{i\theta} - 1}{\Delta x}, \qquad \lambda^2 = -\frac{4\sin^2(\theta/2)}{(\Delta x)^2} = \left(\frac{2i\sin(\theta/2)}{\Delta x}\right)^2.$$

As we have previously seen $\frac{\epsilon}{\lambda} = 1 + O(h)$ and so A and B are first order accurate but λ is second order accurate. The pre-processing stage involves

$$V_0 = \frac{\epsilon}{\lambda} Z_0 \Rightarrow \lambda V_0 = \epsilon Z_0 \tag{19}$$

or in physical space at time $t = 0$,

$$v_{j+1/2}(0) - v_{j-1/2}(0) = z_j(0) - z_{j-1}(0).$$

By inspection this implies $z_j(0) = v_{j+1/2}(0)$. As previously noted we have pre-processed v to get u to the correct order. In order to find v to second order we need to post-process the solution and so we need to translate (18) into physical space. At time t we calculate y from z (y and z are the back Fourier transforms of Y and Z) via

$$y_{j+1/2}(t) - y_{j-1/2}(t) = z_j(t) - z_{j-1}(t)$$

or $y_{j+1/2}(t) = z_j(t)$.

Therefore, to fix Scheme II we merely shift v by half a cell width for both the initial condition and the final evaluation. By linearity it is sufficient to only change these two time levels. Hence, we rederive mathematically what we previously observed intuitively.

We now consider the harder case of Schemes III and IV where the amplitude error reduces from fourth order accurate to first order accuracy. Now

$$\epsilon = \frac{2i\sin(\theta/2)}{\Delta x(\gamma e^{i\theta} + 1 - \gamma)}, \qquad \mu = \frac{2i\sin(\theta/2)}{\Delta x(\gamma e^{-i\theta} + 1 - \gamma)}$$

$$\lambda^2 = -\frac{4\sin^2(\theta/2)}{(\Delta x)^2(1 - 2\beta\sin^2(\theta/2))}.$$

Note that λ is not the symbol of a finite difference operator. As before γ and β are first order accurate. We correct this by pre-processing

$$V_0 = \frac{\epsilon}{\lambda} Z_0 \Rightarrow \lambda V_0 = \epsilon Z_0$$

or

$$\frac{2i \sin(\theta/2)}{\sqrt{(1 - 2\beta \sin^2(\theta/2))}} V_0 = \frac{2i \sin(\theta/2)}{(\gamma e^{i\theta} + 1 - \gamma)} Z_0.$$

Cancelling and clearing fractions we get

$$\sqrt{\left(1 - 2\beta \sin^2\left(\frac{\theta}{2}\right)\right)} Z_0 = (\gamma e^{i\theta} + 1 - \gamma) V_0. \tag{20}$$

Because of the square root that appears in this formula we can interpert (20) only as a pseudo-difference equation in physical space. To overcome this difficulty we need to approximate the square root with fourth order accuracy by a Taylor series expansion or a Pade approximation.

We first approximate the square root by a Taylor series to fourth order accuracy and get

$$\left(1 - \beta \sin^2\left(\frac{\theta}{2}\right) + \beta^2 \sin^4\left(\frac{\theta}{2}\right)\right) V_0 = (\gamma e^{i\theta} + 1 - \gamma) Z_0$$

or in physical space, $v = v_j(t)$

$$v_j(0) - \frac{\beta}{2}(v_{j+1}(0) - 2v_j(0) + v_{j+1}(0)) + \frac{\beta^2}{16}(v_{j+2}(0) - 4v_{j+1}(0)$$
$$+ 6v_j(0) - 4v_{j-1}(0) + v_{j-2}(0)) = \gamma z_{j+1}(0) + (1 - \gamma)z_j(0). \tag{21}$$

This can be solved directly for z_j at the initial time. When we reach the post-processing stage, at $t = N\Delta t$, we have z_j^N and $u_j^N = w_j^N$ at all the grid points and need to calculate y_j^N. Repeating the argument leading to (20) and (21) we obtain

$$y_j(t) - \frac{\beta}{2}(y_{j+1}(t) - 2y_j(t) + y_{j+1}(t)) + \frac{\beta^2}{16}(y_{j+2}(t) - 4y_{j+1}(t)$$
$$+ 6y_j(t) - 4y_{j-1}(t) + y_{j-2}(t)) = \gamma z_{j+1}(t) + (1 - \gamma)z_j(t). \tag{22}$$

However, since now we need to solve for y_j^N in terms of z_j^N, we need to invert a pentadiagonal matrix. Since this is done only whenever the solution is needed the additional cost is not very large. A better way is to replace the square root in (20) by a Pade approximation to fourth order rather than a Taylor Series. We can then use either a tridiagonal solver or even give y explicitly in terms of z with fourth order accuracy depending on the numerator of the Pade approximation. In either case we have now pre- and post-processed the approximation so that both the amplitude and the phase are fourth order accurate.

4. TWO DIMENSIONS

In two dimensions we generalize (1) to a system that is equivalent to the wave equation, namely

$$p_t = q_x + r_y$$
$$q_t = p_x \qquad (23)$$
$$r_t = p_y.$$

We denote the Fourier transform of p, q, r by P, Q, R with Fourier variables k_x, k_y. As before we introduce a numerical scheme and Fourier transform it. We denote the Fourier transform of the numerical quantities by U, V, W. Then we write the Fourier transform of the numerical scheme in the abstract form

$$U_t = \epsilon_x V + \epsilon_y W$$
$$V_t = \mu_x U \qquad (24)$$
$$W_t = \mu_y U,$$

where, as before, the ϵ's and μ's are symbols of first difference operators. Define $\lambda_x^2 = \epsilon_x \mu_x$, $\lambda_y^2 = \epsilon_y \mu_y$, $\lambda^2 = \lambda_x^2 + \lambda_y^2$, and $\sigma^2 = -(k_x^2 + k_y^2)$. Then

$$P_{tt} = \sigma^2(P_{xx} + P_{yy}), \qquad U_{tt} = \lambda^2(U_{xx} + U_{yy}). \qquad (25)$$

We shall assume that all the ϵ and μ are low order approximations to the appropriate k_x, k_y. However, λ_x is a high order approximation to ik_x and similarly in the y direction. The solutions of (23) and (24) are given by

$$P = A_e e^{\sigma t} + B_e e^{-\sigma t}, \qquad U = A e^{\lambda t} + B e^{-\lambda t}$$

$$A_e = \frac{1}{2}\left(P_0 + \frac{ik_x}{\sigma}Q_0 + \frac{ik_y}{\sigma}R_0\right), \qquad A = \frac{1}{2}\left(U_0 - \frac{\epsilon_x}{\lambda}V_0 + \frac{\epsilon_y}{\lambda}W_0\right) \qquad (26)$$

$$B_e = \frac{1}{2}\left(P_0 - ik_x Q_0 - ik_y R_0\right), \qquad B = \frac{1}{2}\left(U_0 - \epsilon_x V_0 + \epsilon_y W_0\right).$$

For the other variables we have

$$Q = \frac{ik_x}{\sigma}\left[A_e(e^{\sigma t} - 1) - B_e(e^{-\sigma t} - 1)\right] + Q_0$$

$$R = \frac{ik_y}{\sigma}\left[A_e(e^{\sigma t} - 1) - B_e(e^{-\sigma t} - 1)\right] + R_0$$

and

$$V = \frac{\mu_x}{\lambda}[A(e^{\lambda t} - 1) - B(e^{-\lambda t} - 1)] + V_0$$

$$W = \frac{\mu_y}{\lambda}[A(e^{\lambda t} - 1) - B(e^{-\lambda t} - 1)] + W_0.$$

Comparing the solutions for P and U we need to choose, for given Q_0, R_0 the initial data (V_0, W_0) so that

$$i\frac{k_x Q_0 + k_y R_0}{\sigma} = \frac{\epsilon_x V_0 + \epsilon_y W_0}{\lambda}. \tag{27}$$

However, ik_x, ik_y are symbols of the continuous operator and not useful for the discrete problem. By assumption, $ik_x \sim \lambda_x$ to the order of the scheme. One way to satisfy (27) to fourth order accuracy is to require (compare with (19))

$$\lambda_x Q_0 = \epsilon_x V_0, \qquad \lambda_y R_0 = \epsilon_y W_0. \tag{28}$$

We define (V, W) as the solution of (24) with the new initial conditions, (28). With this choice $A \sim A_e$, $B \sim B_e$ and so $U \sim P$ to the order of accuracy of the phase error. In order to match V, W with Q, R, to higher order, we must post-process (V, W) so that

$$\begin{aligned}\frac{\epsilon_x}{\lambda_x}V &= \frac{\lambda_x}{\lambda}[A(e^{\lambda t} - 1) + B(e^{-\lambda t} - 1)] + \frac{\epsilon_x}{\lambda_x}V_0 \\ &\sim \frac{\lambda_x}{\lambda}[A(e^{\lambda t} - 1) + B(e^{-\lambda t} - 1)] + Q_0.\end{aligned} \tag{29}$$

However, again recalling that $\lambda \sim \sigma$ and $\lambda_x \sim ik_x$ to higher order, we have $\frac{\epsilon_x}{\lambda_x}V \sim Q$. Hence, we introduce new variables which postprocess (U, V). These are given by

$$\lambda_x \hat{V} = \epsilon_x V, \qquad \lambda_y \hat{W} = \epsilon_y W. \tag{30}$$

By the above analysis we have shown that $\hat{V}$ approximates Q and $\hat{W}$ approximates R to the higher order accuracy that λ approximates σ.

We conclude with the following:

- Without pre- or post-processing we have that $\lambda \sim \sigma$ to the order of the scheme. Hence, the original scheme (24) with the initial conditions P_0, Q_0, R_0 yields a solution with the proper higher order phase error.

- In order to also obtain the amplitude to the correct order of accuracy we need to only pre-process the initial conditions and post-process the numerical solution for V and W but not U. For linear problems we only need post-process at the conclusion of the computation not after each time step. This pre- and post-processing recovers the full accuracy of the scheme.

- The symbols ϵ, μ, λ may not correspond to any finite difference operator. Therefore, to translate these conditions into physical space we may need to approximate these symbols by a Taylor series expansion or a Pade approximation as one done in one dimension.

5. CONCLUSION

We have shown that Schemes II and IV have the same phase error as Schemes I and III, respectively, and so the phase is higher order accurate. However, the change from central differences to one sided differences reduces the amplitudes to first order accuracy. We traced this seeming paradox to the treatment of the initial data. We have further shown that one can recover the higher order accuracy of the scheme by pre- and post-processing the data. In

some cases this processing leads to Fourier symbols that can no longer be simply related to finite difference formulas in physical space. In these cases it is necessary to approximate the symbols by a Taylor series expansion or a Pade approximation to the order of the schemes. Then the total scheme in physical space has the desired order.

ACKNOWLEDGMENTS

REFERENCES

1. R. Janaswany and Y. Liu, An unstaggered colocated finite-difference scheme for solving time-domain Maxwell's equation in curvilinear coordinates, *IEEE Trans. Antennas Propagation* **45**, 1584 (1997).

2. K. S. Kunz and R. J. Luebbers, *Finite Difference Time Domain Method for Electromagnetics* (CRC Press, Boca Raton, FL, 1993).

3. Y. Liu, Fourier analysis of numerical algorithms for the Maxwell equations, *J. Comput. Phys.* **124**, 396 (1996).

4. A. Taflove, *Computational Electrodynamics—The Finite-Difference Time-Domain Method* (Artech House, Norwood, MA, 1995).

5. E. Turkel, High order methods, in *Advances in Computational Electrodynamics: The Finite-Difference Time-Domain Method*, edited by A. Taflove (Artech House, Boston, 1998), p. 63.

6. E. Turkel and G. Zwas, Explicit large time step schemes for the shallow water equations, *Adv. Comput. Methods Partial Differential Equations* **3**, 65 (1979).

7. K. S. Yee, Numerical solution of initial boundary value problems involving Maxwell's equation in isotropic media, *IEEE Trans. Antennas Propagation* **14**, 302 (1966).

8. A. Yefet and E. Turkel, Fourth order compact implicit method for the Maxwell equations with discontinuous coefficients, *Appl. Numer. Math.*, in press.